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THE MOTION OF POINT DEFECTS IN SOLIDS*

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Irreversible deformations in solids (plasticity and creep) are explained, within the framework of the theory of elasticity, by the motion of line and point defects which are always present in the structure of a solid. The dislocations and cracks are regarded as line defects, hole-type defects (vacancies, pores, and cavities), and inclusion-type defects (imbedded atoms, foreign particles, and protons) are regarded as point defects.

The method of invariant integrals represents the most universal method of determining the forces driving these defects. A very brief survey of the literature is given first, followed by a discussion of inclusion-type point defects modelled by centres of dilatation. A formula for the driving force is obtained and certain interactions and motions of the inclusions are studied. After this, hole-type defects are discussed, modelled by the asymptotic to a spherical cavity induced by an external field. A formula for the driving force is given and certain interactions and displacements of the holes are studied. The behaviour of a hole is found to be qualitatively different from that of a model vacancy adopted in the physical literature and described in terms of centres of compression.

In 1951, Eshelby extended Maxwell's method of the theory of the electromagnetic field to obtain three invariant integrals of the theory of elasticity /1/ known later as J-integrals. However, neither Eshelby nor his successors could overcome the fundamental difficulty of the theory connected with the divergence of the invariant integrals at field singularities. Eshelby himself used the old energy formalism based on the determination of the interaction energy (see in /2/) his theory of inclusions or the derivation of the Peach-Koehler formula for the force driving the dislocation). It is for this reason that later investigators ignored J-integrals; numerous books on dislocation theory (e.g. /3-7/) do not even mention them. Wide use of J-integrals began with the paper by Rice in 1968 /8/ in which he used the divergence theorem to prove directly the invariance of one of the J-integrals and used it to analyse the concentration of deformations near notches and cracks. In 1972 Landes and Begley used this to introduce a new constant J_c as a characteristic of the onset of crack growth in elastoplastic materials /9/.

In 1967 the author proposed, independently of the work done By Eshelby, another, more general, approach /10/ enabling the invariant energy Γ-integral to be derived for any solid, taking the dynamic and volume forces into account (the J-integral can be obtained from it as a special case). In this approach the invariance of the Γ-integral becomes a trivial consequence of the law of conservation (using this invariance the author determined in /10,11/ the force Γ driving the crack with the help of various contours, namely of a circumference and a rectangle). Moreover, a general theory of fracture of solids was proposed in /10/; the corresponding fundamental fracture constant was denoted by γ (the constant J_c introduced five years later is equal to 2γ). The approach was further developed by the author using numerous examples in /12-16/, in particular the theory of Γ-residues and the rule for Γ-integration were derived, enabling the divergent invariant integrals to be evaluated and hence enabling one to "work" with the singularities of a physical field.

Below, a homogeneous isotropic elastic medium is considered under quasistatic conditions, when the deformations are small. In this case the law of conservation of energy can be written in the form

$$\Gamma_k = \int_{\Sigma} (U n_k - \sigma_{ij} u_{i,k} n_j)\, d\Sigma \quad (i,\ j,\ k = 1,\ 2,\ 3) \tag{0.1}$$

Here U is the elastic potential per unit volume, $\sigma_{i,j}$ are the stresses, u_i are the displacements, Γ_k are the components of the driving force (equal to the energy dissipated when the singularities of the elastic field within Σ are displaced by unit length along the axis x_k).

The motion of linear defects of dislocation and crack type is caused by the forces given by the Peach-Koehler and Irwin formulas, respectively; the formulas are derived with the help of (0.1) and the rule of Γ-integration by contracting Σ into a singularity /10-16/. The theory of point defects is given below. As regards the terminology, the term "invariant"

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introduced by the author is equivalent to the term "independent of the integration path" used by Rice.

1. Inclusion-type point defects. Examples of inclusion-type point defects are e.g. *N* or *H* atoms pentrating the metal matrix during smelting, atoms of alloying elements dissolved intentionally in metal in order to impart the necessary properties to it, impurity atoms, etc. They can move relative to the metal matrix by selfdiffusion, or under the action of external loads.

We shall model inclusion-type defects centres of dilation described as follows /17/:

$$u_R{}^s = \frac{1+\nu}{2E}\frac{qa^3}{R^2} \quad (q>0) \tag{1.1}$$
$$\sigma_R{}^s = -\frac{qa^3}{R^3}, \quad \sigma_\psi{}^s = \sigma_\theta{}^s = \frac{1}{2}\frac{qa^3}{R^3}$$

Here R, ψ, θ are spherical coordinates, $u_R{}^s, \sigma_R{}^s, \sigma_\psi{}^s, \sigma_\theta{}^s$ denote the non-zero displacement and stresses, a is the radius of the centre of compression "core", q is the pressure exerted by the core on the elastic body, E is Young's modulus and ν is Poisson's ratio.

Using (0.1) and (1.1) we can show that in the case of an arbitrary homogeneous external stress field, the action of the driving force on the inclusion is equal to zero. Let us now suppose that the unperturbed elastic field has the following form (A is a given constant):

$$\sigma_{33}{}^0 = Ax_1, \quad u_1{}^0 = -{}^1/{}_2 A\nu E^{-1}(x_1{}^2 - x_2{}^2 + \nu^{-1}x_3{}^2) \tag{1.2}$$
$$u_2{}^0 = -\nu AE^{-1}x_1x_2, \quad u_3{}^0 = AE^{-1}x_1x_3$$

We will calculate the driving force for this case (directed along the x_1 axis, since its components along the x_2 and x_3 axes are obviously zero). Using the invariance of the Γ-integral (0.1) we take as Σ a paralellepiped formed by the edges $x_3 = \pm\delta$, $x_1 = \pm L$, $x_2 = \pm L$ with $\delta/L \to 0$, $\delta \to \infty$, $L \to \infty$.

According to the rules of Γ-integral we obtain

$$\Gamma_1 = -2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\sigma_{33}{}^s u_{3,1}^0 + \sigma_{33}{}^0 u_{3,1}^s + \sigma_{23}{}^s u_{2,1}^0 + \sigma_{13}{}^s u_{1,1}^0)\,dx_1\,dx_2 \quad (x_3 = \delta) \tag{1.3}$$

Here we have made use of the symmetry about the plane $x_3 = 0$, and the relations $n_3 = 1$ when $x_3 = +\delta$, $n_3 = -1$ when $x_3 = -\delta$, $\sigma_{13}{}^0 = \sigma_{23}{}^0 = 0$ when $x_3 = \pm\delta$.

According to (1.1) we have

$$\sigma_{33}{}^s = \sigma_z{}^s = -\frac{1}{4}\frac{qa^3}{R^3}(1 + 3\cos 2\psi) \tag{1.4}$$
$$\tau_{rz}^s = -\frac{3}{4}\frac{qa^3}{R^3}\sin 2\psi, \quad \sigma_{13} = \frac{x_1}{r}\tau_{rz}, \quad \sigma_{23} = \frac{x_2}{r}\tau_{rz}$$
$$u_{3,1}^s = -3\frac{1+\nu}{2E}qa^3\frac{x_1z}{R^5}\,(z = x_3,\ r^2 = x_1{}^2 + x_2{}^2,\ R^2 = r^2 + z^2)$$

Here r, θ, z are cylindrical coordinates and the angle ψ is measured from the z axis.

Using the relations (with $x_3 = \delta$)

$$R^2 = r^2 + \delta^2,\ \delta/R = \cos\psi,\ r/R = \sin\psi,\ x_1/r = \cos\theta$$
$$r\,dr = R\,dR = R^2\,\mathrm{tg}\,\psi\,d\psi,\ dx_1dx_2 = r\,dr\,d\theta$$

we obtain (1.3) using (1.4) and (1.2)

$$\Gamma_1 = -2E^{-1}A\int_0^\infty\int_0^{2\pi}(\delta\sigma_z{}^s - \nu r\tau_{rz}^s + Ex_1u_{3,1}^s)\,r\,dr\,d\theta = \tag{1.5}$$
$$\pi E^{-1}Aqa^3\int_0^{\pi/2}[\sin\psi(1 + 3\cos 2\psi) - 3\nu\sin\psi\,\mathrm{tg}\,\psi\sin 2\psi +$$
$$3(1+\nu)\sin^3\psi]\,d\psi = 2\pi E^{-1}(1-\nu)\,Aqa^3$$

In the other special case of an inhomogeneous external field

$$\sigma_{11}{}^0 = Bx_1,\ \sigma_{13}{}^0 = \sigma_{31}{}^0 = -Bx_3 \tag{1.6}$$
$$u_1{}^0 = {}^1/{}_2\,E^{-1}B\,[x_1{}^2 + \nu x_2{}^2 - (2+\nu)x_3{}^2]$$
$$u_2{}^0 = -\nu E^{-1}Bx_1x_2,\ u_3{}^0 = -\nu E^{-1}Bx_1x_3$$

analogous calculations yield

$$\Gamma_1 = 2\pi(1-\nu)E^{-1}Bqa^3,\ \Gamma_2 = \Gamma_3 = 0$$

We shall also quote the results of the calculations for the following two cases of an inhomogeneous external field (G is the shear modulus, and C and D are constants):

$$\sigma_{23}^0 = Cx_2,\ \sigma_{13}^0 = -Cx_1,\ u_1^0 = u_2^0 = 0 \quad (1)$$
$$u_3^0 = {}^1/_2 G^{-1}C(x_2^2 - x_1^2)$$
$$\sigma_{23}^0 = Dx_1,\quad u_1^0 = -{}^1/_2 G^{-1}Dx_2x_3 \quad (2)$$
$$u_2^0 = {}^1/_2 G^{-1}Dx_1x_3,\quad u_3^0 = {}^1/_2 G^{-1}Dx_1x_2$$

In both the above cases we have $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$.

According to the basic principle of Γ-integration /10-14/ the driving force is equal to zero also when the external stresses are power functions x_1, x_2, x_3 of second and higher order.

The general case of an arbitrary inhomogeneous external stress field $\sigma_{ij}^0(x_1, x_2, x_3)$ is obtained, as can be shown, by linear superposition of the basic special cases (after expanding the function σ_{ij}^0 in a Taylor series).

The final general result is

$$\Gamma_k = \lambda_1 E^{-1}\Delta\partial\sigma/\partial x_k\ (k = 1, 2, 3) \quad (1.7)$$
$$\Delta = qa^3,\ \sigma = \sigma_{11}^0 + \sigma_{22}^0 + \sigma_{33}^0,\ \lambda_1 = 2\pi(1-\nu)$$

Thus the driving force acting on the inclusion is directly proportional to the gradient of the first invariant of the external stress tensor $\sigma_{ij}^0(x_1, x_2, x_3)$. Formula (1.7) is an analogue of the Peach-Koehler formula (in the theory of dislocations) and of the Irwin formula (in the theory of cracks). It can also be derived from expression (8.9) of the first equation of Sect.b, par.5 of /2/.

Using (1.7) we will consider several basic problems.

Interaction between the inclusions. Let one of the inclusions be situated at the origin of coordinates. According to (1.1), $\sigma = 0$ in the stress field generated by it, and therefore from (1.7) we conclude that the inclusions do not interact with each other.

Interaction between the inclusions and an edge dislocation. This problem was discussed in /3/ using the energy method. The results are naturally the same, but we see that the force approach is much simpler. Let the line of an edge dislocation coincide with the x_3 axis, and let its Burgers vector B be directed along the x_1 axis. The quantity σ for such a dislocation will be equal to

$$\sigma = \frac{BEx_2}{2\pi(1-\nu)(x_1^2 + x_2^2)} \quad (1.8)$$

Let us place an inclusion at the point (x_1, x_2) and calculate the driving force acting on it, using Eqs.(1.7) and (1.8)

$$\Gamma_1 = -\frac{\lambda_1\Delta Bx_1x_2}{\pi(1-\nu)(x_1^2+x_2^2)^2},\quad \Gamma_2 = \frac{\lambda_1\Delta B(x_1^2 - x_2^2)}{2\pi(1-\nu)(x_1^2+x_2^2)^2} \quad (1.9)$$

The trajectory of the mobile inclusion is a solution of the equation

$$dx_2/dx_1 = \Gamma_2/\Gamma_1 = (x_2^2 - x_1^2)/(2x_1x_2) \quad (1.10)$$

The general solution of (1.10) has the form

$$x_1^2 + x_2^2 = Cx_1\ (C = \text{const}) \quad (1.11)$$

The family (1.11) represents a set of circles touching the x_2 axis at the origin of coordinates (with the centres lying on the x_1 axis); according to (1.9) the motion of the inclusion along the circle is anticlockwise when $x_1 > 0$ and $B > 0$, and clockwise when $x_1 < 0$. Thus the mobile inclusions are attracted to the core of the edge dislocation from the side of extension.

Using the invariance of the Γ-integral, we can prove the following law of action and reaction; if a certain singularity A brings about a configurational force $\mathbf{\Gamma}$ acting on another singularity B, then a configurational force $\mathbf{\Gamma}$ equal in magnitude and opposite in direction will act from the singularity B on the singularity A.

If a smooth accumulation of inclusions is distributed within the material, then the force Γ_1 driving the edge dislocation, according to (1.9) and the law of action and reaction, will be equal to

$$\Gamma_1 = B\tau_\infty + \frac{\lambda_1 \Delta B}{\pi(1-\nu)} \iint \frac{x_1 x_2 N(x_1, x_2)\, dx_1 dx_2}{(x_1^2 + x_2^2)^2} \tag{1.12}$$
$$\left(\delta = \frac{4}{3}\pi a^3 N \rho_{inc}\right)$$

Here τ_∞ is the external shear stress in the plane $x_2 = 0$, N is the number of inclusions per unit volume, ρ_{inc} is the inclusion density and δ is the mass of the inclusions per unit volume. The first term in (1.12) is the Peach-Koehler force, and the second term represents the force induced by the accumulation of inclusions.

Interaction between inclusions and a crack. Let an inclusion be situated in the neighbourhood of the front of a crack of normal discontinuity $x_2 = 0,\ x_1 < 0,\ -\infty < x_3 < +\infty$. The polar coordinates of the inclusion are r and θ (in the plane $x_3 = 0$). In this case we have (K_I is the stress intensity factor)

$$\sigma = \sqrt{2/\pi}\,(1 + \nu)\, K_I r^{-1/2} \cos(\theta/2) \tag{1.13}$$

Using (1.7) and (1.13) we find the components of the driving force in polar coordinates

$$\Gamma_r = -\frac{\lambda_1 \Delta (1+\nu)}{E\sqrt{2\pi}} K_I \frac{\cos\theta/2}{r^{3/2}}, \quad \Gamma_\theta = -\frac{\lambda_1 \Delta (1+\nu)}{E\sqrt{2\pi}} K_I \frac{\sin\theta/2}{r^{3/2}} \tag{1.14}$$

The trajectory of the mobile inclusion is, according to (1.14), a solution of the equation

$$r d\theta/dr = \Gamma_\theta/\Gamma_r = \operatorname{tg}\theta/2$$

and this implies that the family of trajectories (cardioids) will be

$$\sqrt{r} = C \sin\theta/2 \quad (C = \text{const})$$

The curves represent closed ovals symmetrical about the x_1 axis, and have an internal cusp at the origin of coordinates; the inclusions move clockwise when $x_2 > 0$, and counter clockwise when $x_2 < 0$. Thus the inclusions are attracted to the crack tip along its continuation (i.e. in the region where fracture has not yet occurred).

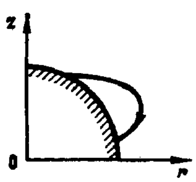

Fig.1

According to the law of action and reaction (1.14) implies that the driving force of the crack front, when there is a continuous accumulation of stationary inclusions, is equal to

$$\Gamma_1 = \frac{1-\nu^2}{E} K_I^2 + \frac{\lambda_1 \Delta (1+\nu) K_I}{E\sqrt{2\pi}} \iint \frac{N(r,\theta)}{r^{3/2}} \cos\frac{3\theta}{2}\, dr\, d\theta \tag{1.15}$$
$$(\Gamma_1 = \Gamma_r \cos\theta - \Gamma_\theta \sin\theta)$$

The first term in (1.15) is the Irwin force (due to the external load) and the second term represents the force induced by the accumulation of inclusions.

Interaction between an inclusion and a spherical cavity. Let an infinite space with a spherical cavity of radius R_a be stretched uniaxially by the stress $\sigma_z = p$. The centre of the sphere coincides with the cylindrical rz origin of coordinates, and the surface of the cavity is free of external loads. In this case the sum of the normal stresses σ in an elastic body will be equal to /17/

$$\sigma = p + \frac{5p(1+\nu)}{7-5\nu} R_a^3 \frac{\partial}{\partial z}\frac{z}{R^3} = p + 5pR_a^3 \frac{1+\nu}{7-5\nu}\frac{r^2 - 2z^2}{R^5} \tag{1.16}$$
$$(R^2 = r^2 + z^2)$$

Let an inclusion be present at some point of the body, and let a driving force with components

$$\Gamma_r = \frac{15\lambda_1 \Delta p (1+\nu)}{E(7-5\nu)} \frac{rR_a^3(4z^2 - r^2)}{R^7}$$
$$\Gamma_z = \frac{15\lambda_1 \Delta p (1+\nu)}{E(7-5\nu)} \frac{zR_a^3(2z^2 - 3r^2)}{R^7}$$

act, according to (1.7) and (1.16), on this inclusion. The velocity of the moving inclusion has the same direction as the driving force, therefore the trajectory of the inclusion is an integral curve of the following equation:

$$\frac{dr}{dz} = \frac{r(4z^2 - r^2)}{z(2z^2 - 3r^2)}$$

Fig.1 shows, in a qualitative manner, the trajectories of motion of the inclusions. We see that the inclusions move into the most stressed zone of tensile stresses near the cavity (the hardening effect of the inclusions).

Continuum theory of inclusions. Let a rigid body contain a very large number of small inclusions (a cloud or accumulation. In the asymptotic approximation discussed here the inclusions can be regarded as non-interacting. The external stresses will be assumed to be fairly small, so that the motion of the inclusions will be subcritical.

The rate of drift of the inclusions $\mathbf{v}_d$ will be assumed (as is usually done in the linear theory of diffusion), to be directly proportional to the driving force $\mathbf{\Gamma}$ /12, 18/, i.e. $\mathbf{v}_d = \eta\mathbf{\Gamma}$ where η is the empirical coefficient of the mobility of inclusions. Here the inclusion mass transport equation will have the form

$$\partial\delta/\partial t = (D\delta_{,k})_{,k} - (\eta E^{-1}\lambda_1\Delta\delta\sigma_{,k})_{,k}\ (k = 1, 2, 3) \tag{1.17}$$

Here D is the selfdiffusion coefficient of the inclusions, t is the time and $\delta(t, x_1, x_2, x_3)$ is the mass of the inclusions per unit volume. The constants η, Δ and D depend on the temperature of the body.

Eq.(1.17) can be written for inclusions of one and the same type, with the same coefficients D, η and Δ. The total number of equations is equal to the number of types of inclusions.

The effect of other external fields (e.g. an electric, thermal or chemical field) on the motion of the inclusions is studied using well-known methods /12, 18/, and additional terms appear /19/ here in Eq.(1.17).

Let us give the simplest example of the solution of (1.17). Let the strip $0 < x_1 < d$ be subjected to pure bending by the moment M (per unit length). Then, from the elastic solution it follows that $\sigma_{,2} = \sigma_{,3} = 0$, $\sigma_{,1} = -12(1+\nu)Md^{-3}$. The equilibrium concentration of the inclusions established in the strip as a result of flexure is determined by the solution of (1.17) when $\partial\delta/\partial t = 0$. Moreover, when the mobile inclusions are in equilibrium, so that the inclusion flux is zero, the condition $\delta = 0$ when $x_1 = d$, must obviously hold, since physical arguments imply that $\delta \geqslant 0$ always. We find

$$\delta = -C_0(e^{-\mu d} - e^{-\mu x_1}),\quad \mu = \frac{12(1+\nu)\eta\lambda\Delta M}{EDd^3} \tag{1.18}$$

The constant C_0 and be found if the total mass m_0 of mobile inclusions is known. We have

$$C_0 = \mu m_0[1 - (1 + \mu d)e^{-\mu d}]^{-1}\left(\int_0^d \delta\, dx_1 = m_0\right)$$

The solution (1.18) makes possible e.g. the prediction of the distribution of material properties across the strip thickness, provided that the dependence of the corresponding property on the concentration of inclusions is known.

2. Hole-type point defects. Hole-type point defects include vacancies in a crystal lattice, various pores and cavities. For sufficiently small sizes (or when the material structure has channels), the defects can move relative to the lattice under the action of an external load and by selfdiffusion.

We shall model the hole-type defects by a spherical cavity of radius r_0, whose surface is free from external loads. We shall assume that the distance R between the holes is much larger than r_0 (in practice, $R > 4r_0$ is sufficient, i.e. the relative porosity of the material in question should be less than 0.07). Under this assumption the hole becomes a source of an asymptotically singular perturbation (of order $O(R^{-2})$ in displacements) and it can be regarded as some "quasiparticle" /12-16/. A driving (configurational) force $\mathbf{\Gamma}(\Gamma_1, \Gamma_2, \Gamma_3)$ acts on the hole, as well as on any source of perturbations, and its components are given by Eq. (0.1).

In the physical literature (e.g. /1–7/ the vacancies and micropores are described by Eqs.(1.1) with $q<0$. Thereby the micropore is regarded as a centre of compression with a specified coefficient for a singularity which is independent of the external field. This corresponds to reality only in the case of a uniform volumetric compression q. However, even in this case the coefficient accompanying the singularity qr_0^3 is determined by the external field. It appears that the model of a complex, hole-type singularity induced by a field developed below, is closer to reality.

The displacements u_r and u_z arising when an elastic space with a spherical cavity of radius r_0 is stretched, will be as follows /17/:

$$u_r = -\frac{\nu p}{E}r + \frac{prr_0^3}{4GR^3}\left[-\frac{z^2}{R^2} - \frac{4-10\nu}{7-5\nu}\left(1 - \frac{3z^2}{R^2}\right) + \frac{3r_0^2 - (2+5\nu)R^2}{(7-5\nu)R^2}\left(1 - \frac{5z^2}{R^2}\right)\right] \tag{2.1}$$

$$u_z = \frac{p}{E}z + \frac{pzr_0^3}{4GR^3}\left[2 - \frac{z^2}{R^2} - \frac{4-10\nu}{7-5\nu}\left(1 - \frac{3z^2}{R^2}\right) + \frac{(2+5\nu)R^2 - 3r_0^2}{(7-5\nu)R^2}\left(-3 + \frac{5z^2}{R^2}\right)\right]$$

$$(R^2 = r^2 + z^2)$$

Here r, z are cylindrical coordinates with origin at the centre of the sphere, and p is the value of tensile stress σ_z (along the z axis).

The stresses can be found from (2.1) using Hooke's law /20/.

We shall use the invariance of Γ_k relative to Σ, and contract Σ to the surface of the spherical cavity. From this we see, using (0.1) and (2.1), that $\Gamma_k = 0$. The following general result can also be shown: in the case of an arbitrary homogeneous external field, the driving force acting on a strip of arbitrary shape is equal to zero (the analgoue of the d'Alembert-Euler paradox of hydrodynamics).

Let us now assume that the external (unperturbed) field still has an inhomogeneous component $\sigma_{33}{}^0 = Ax_1$, and $Ar_0 \ll p$; and that the following displacement field corresponds to it:

$$u_1{}^0 = -{}^1/_2\, \nu A E^{-1}\left(x_1{}^2 - x_2{}^2 + \frac{1}{\nu} x_3{}^2\right), \quad u_2{}^0 = -\nu A E^{-1} x_1 x_2. \tag{2.2}$$
$$u_3{}^0 = A E^{-1} x_1 x_3 \; (x_3 = z,\; x_1{}^2 + x_2{}^2 = r^2)$$

Using the invariance of the Γ-integral (0.1) we shall use the parallelepiped $x_3 = \pm\delta$, $x_1 = \pm L$, $x_2 = \pm L$ with $\delta/L \to 0$, $\delta \to \infty$, $L \to \infty$ as Σ. This yields, according to the rules of Γ-integration /12-16/

$$\Gamma_1 = -2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (\sigma_{33}^0 u_{3,1}^s + \sigma_{33}^s u_{3,1}^0 + \sigma_{23}^s u_{2,1}^0 + \sigma_{13}^s u_{1,1}^0)\, dx_1\, dx_2 \tag{2.3}$$

with $x_3 = \delta \to \infty$.

The singular perturbation field (with index s) is determined using Eqs.(2.1). Calculations analogous to those of Sect.1, yield the value of the driving force

$$\Gamma_1 = \lambda_2 E^{-1} p A r_0{}^3 \; (\Gamma_2 = \Gamma_3 = 0) \tag{2.4}$$
$$\lambda_2 = \frac{10\pi}{7-5\nu}\left[12\nu^2 - 4\nu + \frac{\pi(1+\nu)}{160}(123 - 180\nu)\right]$$

We see that the driving force is directly proportional to the external stress and its gradient.

Let us now consider the general case of an arbitrary inhomogeneous external stress field $\sigma_{ij}{}^0(x_1, x_2, x_3)$, satisfying the condition $|\sigma_{ij}{}^0| \gg r_0 |\sigma_{ij,k}{}^0|$. The coefficients accompanying the hole-type singularities will, in this case, be directly proportional to the stresses $\sigma_{ij}{}^0$ at the site of the hole, and the driving force, as implied by the rules of Γ-integration, will be directly proportional to the stresses $\sigma_{ij}{}^0$ and their gradients $\sigma_{ij,k}{}^0$ at the same position. Consequently, the energy of interaction between the hole and the external stress field will be a quadratic function of the stresses. In the general case of an anisotropic body and holes of arbitrary configuration, it will be equal to

$$U = C_{ijmn}\, \sigma_{ij}\sigma_{mn} \; (\Gamma_k = -\partial U/\partial x_k) \tag{2.5}$$

where (C_{ijmn} are constants and Γ_k are the components of the driving force.

In the isotropic case (a spherical hole and an isotropic elastic body) the energy U can depend only on the first and second invariant of the stress tensor σ and on I at the given point. Therefore it can be written as follows (α and β are certain constants):

$$U = E^{-1} r_0{}^3 (\alpha\sigma^2 + \beta I) \; (\Gamma_k = -\partial U/\partial x_k) \tag{2.6}$$
$$\sigma = \sigma_{11}{}^0 + \sigma_{22}{}^0 + \sigma_{33}{}^0$$
$$I = \sigma_{11}{}^0\sigma_{22}{}^0 + \sigma_{22}{}^0\sigma_{33}{}^0 + \sigma_{11}{}^0\sigma_{33}{}^0 - (\sigma_{12}{}^0)^2 - (\sigma_{13}{}^0)^2 - (\sigma_{23}{}^0)^2$$

Let us find the constants α and β. In the case of a uniform volumetric expansion $\sigma_{ij}{}^0 = {}^1/_3\sigma\delta_{ij}$ $(\sigma > 0)$ the hole will obviously behave as a centre of compression (1.1) and the energy of its interaction with the external field, according to (1.7), will be equal to $-\lambda_1\sigma^2 r_0{}^3/(6E)$. This, together with (2.6), yields

$$3\alpha + \beta = -\pi(1-\nu) \tag{2.7}$$

In the other case studied, when $\sigma_{33}{}^0 = p$, $\sigma_{33,1}^0 = A$, and all remaining stresses are zero, the interaction energy, according to (2.4), will be equal to $-\lambda_2 x_1 E^{-1} p A r_0{}^3$, and according to (2.6)$-\,2\alpha E^{-1} p A x_1 r_0{}^3$. Therefore $\alpha = -{}^1/_2\lambda_2$ and taking (2.7) into account we have

$$\alpha = -{}^1/_2\lambda_2,\; \beta = {}^3/_2\lambda_2 - \pi(1-\nu) \; (\alpha < 0,\; \beta > 0) \tag{2.8}$$

Relations (2.5) and (2.6) represent, in the case of a force driving a hole-type singularity induced by the field, an analogue of the Peach-Koehler formula in the theory of dislocations, and of the Irwin formula in the theory of cracks. According to these relations the driving force tries to shift the hole into a more highly stressed zone (irrespective of the sign of the stress). Thus the behaviour of the hole is sometimes different in kind from the behaviour of the model vacancy adopted in the physical literature and described by the Eqs.(1.1) for a given $q<0$. For example, when the state resembles a volumetric expansion, the hole behaves more like an inclusion.

Continuum theory of holes. Let a dense elastic material contain a very large number of small holes, which can be described in terms of a continuous distribution of an accumulation (or a cloud) of holes. The resistance against the motion of a hole depends on the properties of the material, and the size and shape of the hole. We shall assume that the motion of a hole is subcritical and occurs under the action of external stresses and thermal fluctuations (we neglect the interaction between the holes themselves).

A flux of holes is obviously equivalent to a flux of matter (moving in the opposite direction). Therefore the diffusion of the holes leads to the following equation of mass transfer:

$$\partial\varepsilon/\partial t = (D\varepsilon_{,k})_{,k} + (\eta\varepsilon U_{,k})_{,k} \tag{2.9}$$
$$(\varepsilon = (\rho_0 - \rho)/\rho_0,\ \Gamma_k = -U_{,k},\ v_d = \eta\Gamma)$$

Here t is the time, ε is the porosity, $\rho(x_1, x_2, x_3, t)$ is the macroscopic density of the material required, D is the selfdiffusion coefficient of the holes, η is the hole drift coefficient and ρ_0 is the density of the material without holes. The second term on the right-hand side of (2.9) was obtained with the help of (2.5), under the usual assumption that the hole drift rate is directly proportional to the driving force /12/.

We should supplement Eq.(2.9) with the equation of the theory of elasticity

$$\sigma^0_{ij,j} = 0,\ \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1+\nu}{E}\sigma_{ij}^0 - \frac{\nu}{E}\sigma^0_{kk}\delta_{ij} \tag{2.10}$$

For a porous body it is natural to adopt the "rule of mixtures" /15/

$$E = E_{\max}\rho/\rho_{\max} \tag{2.11}$$

The closed system of Eqs.(2.9)–(2.11) enables us to study the evolution of the accumulation (cloud) of holes and the gradual formation of the weakened, crack-line high porosity zones.

Interaction of the holes with the crack front. Let a hole of radius r_0 be situated in some neighbourhood of the crack front of the normal discontinuity $\theta = \pi,\ 0 < r < \infty,\ -\infty < z < \infty$ (r, θ, z are cylindrical coordinates). The stress field will be

$$\begin{aligned}
\sigma_r^0 &= K_I(2\pi r)^{-1/2}(\cos\theta/2 + {}^1/_2\sin\theta\sin\theta/2) \\
\sigma_\theta^0 &= K_I(2\pi r)^{-1/2}(\cos\theta/2 - {}^1/_2\sin\theta\sin\theta/2) \\
\tau_{r\theta}^0 &= K_I(2\pi r)^{-1/2}\sin\theta\cos\theta/2,\ \sigma_z = 2\nu K_I(2\pi r)^{-1/2}\cos\theta/2
\end{aligned} \tag{2.12}$$

The components of the driving force will, according to (2.6), be

$$\begin{aligned}
&\Gamma_r = -\partial U/\partial r,\ \Gamma_\theta = -r^{-1}\partial U/\partial\theta \\
&U = r_0^3K_I^2(2\pi Er)^{-1}\cos^2\theta/2\,[4\alpha(1+\nu)^2 + \beta(4\nu + \cos^2\theta/2)] \\
&\sigma = 2(1+\nu)K_I(2\pi r)^{-1/2}\cos\theta/2,\ I = K_I^{-2}(2\pi r)^{-1}\cos^2\theta/2\,(4\nu + \\
&\quad\cos^2\theta/2)
\end{aligned} \tag{2.13}$$

Using (2.12) and (2.13) (r, θ are the hole coordinates), we obtain

$$\begin{aligned}
\Gamma_r &= r_0^3K_I^2(2\pi Er^2)^{-1}\cos^2\theta/2\,[4\alpha(1+\nu)^2 + \beta(4\nu + \cos^2\theta/2)] \\
\Gamma_\theta &= r_0^3K_I^2(2\pi Er^2)^{-1}\sin\theta\,[2\alpha(1+\nu)^2 + \beta(2\nu + \cos^2\theta/2)]
\end{aligned} \tag{2.14}$$

We see that Γ_r is negative for all θ in the range $0<\theta<\pi$, i.e. the hole is always attracted to the tip of the crack. The trajectory of the mobile hole is shown qualitatively in Fig.2.

The family of trajectories is determined by integrating the equation

$$v_r/v_\theta = d\ln r/d\theta = \Gamma_r/\Gamma_\theta$$

Depending on the relative size of the mobile holes r_0 and the opening of the crack at its tip $2\delta_0$, the flow of the holes towards the crack tip can lead either to retarding the development of the crack, or to reducing the resistance even to subcritical growth of the crack. Moreover, the presence of the holes leads to a change in the driving force of the crack front. Let the cloud of holes be distributed in the material in such a way, that there are N holes per unit volume (the porosity of the material is $\varepsilon = {}^4/_3\pi Nr_0^3$). Summing the forces induced by the external load and the holes, we obtain

$$\Gamma = \frac{1-\nu^2}{E}K_I^2 - \int_{\delta_0}^{L}\int_{-\pi}^{\pi} N(\Gamma_r\cos\theta - \Gamma_\theta\sin\theta)\,r\,dr\,d\theta \tag{2.15}$$

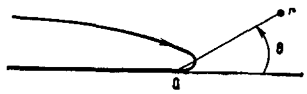

Fig.2

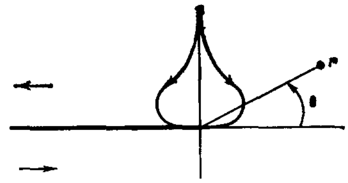

Fig.3

Here Γ is the force driving the holes and L is the characteristic length of the crack (or the body). The first term in (2.15) is the Irwin force (from the external load), and the second term describes the force induced by the cloud of holes.

In the case when the holes are uniformly distributed, N and ε are independent of r and θ and we can use (2.14) to find that the second term of (2.15) is equal to zero.

Interaction between holes and a dislocation. Let a hole of radius r_0 be acted upon by the edge dislocation field

$$\sigma_r^0 = \sigma_\theta^0 = \frac{BE\sin\theta}{4\pi(1-\nu^2)r}, \quad \tau_{r\theta}^0 = -\frac{BE\cos\theta}{4\pi(1-\nu^2)r}, \quad \sigma_z^0 = 2\nu\sigma_r^0 \tag{2.16}$$

Here B is the Burgers vector and the remaining notation is as above.

Using (2.13) and (2.16) we calculate the components of the force driving the hole (r, θ are the hole coordinates)

$$\Gamma_r = \frac{B^2Er_0^3}{8\pi^2(1-\nu^2)^2r^3}\{2\sin^2\theta[2\alpha(1+2\nu)+\beta(1+2\nu)]-\beta\} \tag{2.17}$$

$$\Gamma_\theta = -\frac{B^2Er_0^3}{8\pi^2(1-\nu^2)^2r^3}\sin 2\theta\,[2\alpha(1+\nu)^2+\beta(1+2\nu)]$$

We see that the hole is always attracted towards the core of the dislocation.

According to (2.17), the trajectory of the mobile hole is of the form depicted qualitatively in Fig.3.

If the cloud of holes is distributed throughout the materials, then the force Γ driving the edge dislocation will be equal to

$$\Gamma = B\tau_\infty - \int_{r_d}^{L}\int_{-\pi}^{+\pi} rN(\Gamma_r\cos\theta - \Gamma_\theta\sin\theta)\,dr\,d\theta \tag{2.18}$$

Here r_d is the radius of the dislocation core, L is the characteristic length of the body, and τ_∞ is the external shear stress in the plane of the dislocation. The first term is the Peach-Koehler force, and the second term is the force induced by the cloud of holes. When the holes are distributed uniformly, the second term vanishes.

The proposed force-based approach is purely asymptotic, therefore the formulas obtained hold also for the plastic and finite deformations provided that the domain of non-linearity near the pore is small compared with the distance separating the pores from each other, and with the distance between the pore in question and other sources of perturbation /19/.

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VORTICAL FLOWS AND CANONICAL EQUATIONS OF MOTION OF A MAGNETIZABLE, PERFECTLY CONDUCTING FLUID*

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The classical Kelvin's circulation theorem and Helmholtz theory on the motion of vortex lines with the fluid and conservation of the strength of vortex tubes are generalized to the case of the vortical adiabatic flows of a magnetizable, perfectly conducting fluid. Canonical variables are found and canonical Hamiltonian equations of motion are obtained.

1. The equation of motion of the fluid in question has the form**(**Golosov V.V., Vasil'eva N.L., Taktarov N.G. and Shaposhnikova G.A. Hydrodynamic equations for polarizable, magnetizable, multicomponent and multiphase media. Discontinuous solutions. Study of discontinuous solutions with a jump in magnetic permeability. Moscow, Izd-vo MGU, 1975)

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla p + \frac{B_k}{4\pi}\nabla H_k + \left[\mathbf{j}\times\frac{\mathbf{B}}{c}\right]; \tag{1.1}$$
$$p = p_0 + \frac{1}{4\pi}\int\limits_0^H\left[\mu - \rho\left(\frac{\partial\mu}{\partial\rho}\right)_{T,H}\right]\mathbf{H}\,d\mathbf{H}$$

where $\mathbf{j}$ is the electric current density, $\mathbf{B} = \mu(\rho, T, H)\mathbf{H}$ is the magnetic induction, T is the temperature, p_0 is the pressure of the normal fluid without the magnetic field, and the remaining notation is standard. We will use below the Gibbs thermodynamic identities for a magnetizable medium

$$dU = T\,dS + \frac{p}{\rho^2}d\rho + \mathbf{H}d\frac{\mathbf{B}}{4\pi\rho}, \quad dW = T\,dS + \frac{dp}{\rho} + \mathbf{H}d\frac{\mathbf{B}}{4\pi\rho} \tag{1.2}$$
$$U = U_0 + \frac{1}{4\pi\rho}\left\{\mathbf{HB} + \int\limits_0^H\left[T\left(\frac{\partial\mu}{\partial T}\right)_{\rho,H} - \mu\right]\mathbf{H}\,d\mathbf{H}\right.$$
$$W = W_0 + \frac{1}{4\pi\rho}\left\{\mathbf{HB} + \int\limits_0^H\left[T\left(\frac{\partial\mu}{\partial T}\right)_{\rho,H} - \rho\left(\frac{\partial\mu}{\partial\rho}\right)_{T,H}\right]\mathbf{H}\,d\mathbf{H}\right.$$
$$S = S_0 + \frac{1}{4\pi\rho}\int\limits_0^H\left(\frac{\partial\mu}{\partial T}\right)_{\rho,H}\mathbf{H}\,d\mathbf{H}$$

Here U, W, S denote, respectively, the internal energy, enthalpy and entropy per unit mass of the fluid, and the zero subscripts denote the parameters without a magnetic field.

Using relations (1.2) we can write (1.1) in the form

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